

Mathematics Assessment Project
CLASSROOM CHALLENGES
A Formative Assessment Lesson

Interpreting Equations

Mathematics Assessment Resource Service
University of Nottingham & UC Berkeley

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Interpreting Equations

MATHEMATICAL GOALS

This lesson unit is intended to help students to:

- Connect algebraic equations to real-life situations.
- Uncover and address misconceptions concerning the meaning of variables in equations.

COMMON CORE STATE STANDARDS

This lesson relates to the following *Standards for Mathematical Content* in the *Common Core State Standards for Mathematics*:

- 6.EE: Apply and extend previous understandings of arithmetic to algebraic expressions.
Reason about and solve one-variable equations and inequalities.
Represent and analyze quantitative relationships between dependent and independent variables.

This lesson also relates to the following *Standards for Mathematical Practice* in the *Common Core State Standards for Mathematics*, with a particular emphasis on Practices 2, 4, and 7:

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
4. Model with mathematics.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

INTRODUCTION

This lesson unit is structured in the following way:

- Before the lesson, students work individually on an assessment task designed to reveal their current understanding. You then review their responses and create questions for students to consider when improving their work.
- In a whole-class discussion, students consider how the meaning of different equations and expressions changes when the definitions of the variables change.
- Students then work in small groups (pairs or threes) on a collaborative discussion task, finding word explanations to go alongside the appropriate expressions and equations. Students are encouraged to challenge one another.
- In a whole-class discussion, students discuss what they have learned.
- Finally, students revisit their initial work on the assessment task and work alone on a task similar to the introductory task.

MATERIALS REQUIRED

- Each student will need a copy of the assessment tasks *Real-life Equations* and *Real-life Equations (revisited)*.
- Each small group of students will need some plain paper, cut-up copies of the *Statements and Equations* sheets, a large sheet of paper for making a poster, and a glue stick.

TIME NEEDED

15 minutes before the lesson, a 60-minute lesson, and 20 minutes in a follow-up lesson. The timings given are approximate. Exact timings will depend on the needs of your students.

BEFORE THE LESSON

Assessment task: *Real-life Equations* (15 minutes)

Have the students complete this task, in class or for homework, a few days before the formative assessment lesson. This will give you an opportunity to assess the work and to find out the kinds of difficulties students have with it. You should then be able to target your help more effectively in the subsequent lesson.

Give each student a copy of the assessment task *Real-life Equations* and briefly introduce the task:

Answer the questions on the sheet carefully. Make sure that you explain your answers clearly. I want to understand how you are thinking about these questions.

It is important that, as far as possible, students are allowed to answer the questions without assistance. They should not have access to calculators.

Students should not worry too much if they cannot understand nor do everything, because in the next lesson they will work on a similar task that should help them. Explain to students that by the end of the next lesson they should be able to answer questions such as these confidently. This is their goal.

Real-life Equations

1. Suppose that there are some chairs in a room and that each chair has 4 legs.
 x = the **number** of chairs.
 y = the **total number** of legs on all the chairs.

Put a check mark in the box next to every equation below that you think is correct.

(a) $x = 4y$ (b) $y = 4x$ (c) $x = \frac{y}{4}$ (d) $y = \frac{x}{4}$

Explain your answer(s).

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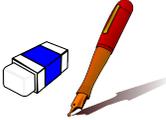
2. Suppose that Max buys some pens and erasers.

The cost of a pen in dollars is x and the cost of an eraser in dollars is y .

p = the **number** of pens he buys.
 e = the **number** of erasers he buys.

The following two equations are true.

Write down the meaning of each equation in words.
 (Refer to pens and erasers in your answers, don't just use letters.)



$p = 2e$	$x = 2y$
In everyday words this means:	In everyday words this means:
.....
.....
.....
.....
.....

Assessing students' responses

Collect students' responses to the task. Make some notes on what their work reveals about their current levels of understanding and their different problem-solving approaches.

We suggest that you do not score students' work. Research shows that this will be counterproductive, as it will encourage students to compare their scores and distract their attention from what they can do to improve their mathematics.

Instead, help students to make further progress by summarizing their difficulties as a series of questions. Some suggestions for these are given in the *Common issues* table on the next page. These have been drawn from common difficulties observed in trials of this unit.

We suggest you make a list of your own questions, based on your students' work. We recommend you either:

- write one or two questions on each student's work, or
- give each student a printed version of your list of questions, and highlight appropriate questions for each student.

If you do not have time to do this, you could select a few questions that will be of help to the majority of students, and write these on the board when you return the work to the student. Students will receive their work and your comments in the follow-up lesson.

Common issues**Suggested questions and prompts**

<p>Selects the wrong equations</p> <p>For example: The student selects the equation $y=4x$ (Q1).</p>	<ul style="list-style-type: none"> • Which is the greater number: the number of chairs or the number of legs? • Say the equation in words.
<p>Selects only one of the two correct equations (Q1)</p>	<ul style="list-style-type: none"> • Is this the only equation that is correct?
<p>Selects equations that are incompatible (Q1)</p> <p>For example: The student selects the first two equations.</p>	<ul style="list-style-type: none"> • Suppose that $y=12$ (there are 12 legs). What do these two equations say about the number of chairs? Which one makes sense?
<p>Misinterprets the meaning of the symbols</p> <p>For example: The student interprets the variables as if they were the objects:</p> <p>The student selects the first equation $x = 4y$ and writes that it means ‘Chair has four legs’ (Q1).</p> <p>For $p = 2e$ the student writes ‘There are two erasers for every pen’ (Q2).</p>	<ul style="list-style-type: none"> • Suppose that $y=12$ (there are 12 legs). What does your equation say about the number of chairs? • Which is greater, x or y? Why? • What does p stand for? What does e stand for?
<p>Substitutes the phrases literally, but does not interpret the meaning in context</p> <p>For example: The student writes ‘Two times e equals p’ or ‘x equals two times y’ (Q2).</p>	<ul style="list-style-type: none"> • What do you know about the costs of the pens and erasers? • What does $2p$ represent? • Can you write the meaning as you would say it in everyday life?
<p>Answers both questions correctly</p> <p>The student needs an extension task.</p>	<ul style="list-style-type: none"> • Can you write an equation which represents the total amount Max spent?

SUGGESTED LESSON OUTLINE

Whole-class introduction (20 minutes)

Begin by showing Slide P-1, which simply contains the question:

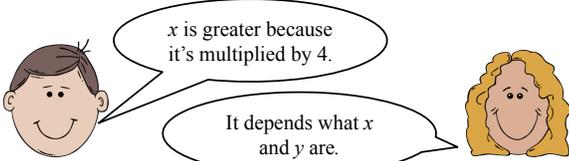
Which is greater, x or y ? Why?

$$y = 4x$$

Ask students to discuss this in pairs.

Then show Slide P-2, which includes two possible answers:

Which is greater, x or y ? Why?

$$y = 4x$$


What do you think? Why? [y is greater because it is four times the size of x.]

Students may be concerned about the possibility of x and y being negative, when words like ‘greater’ can be ambiguous. If this is raised as a problem, you could say that in today’s lesson all our variables will take positive values.

Then show Slide P-3:

What is the equation? Why?

Let e be the **number** of eggs.
Let b be the **number** of egg boxes.
There are 6 eggs in each box.
Find an equation linking e and b .

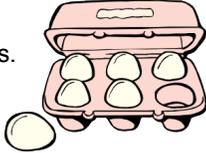


Again, invite students to discuss this with their neighbor.

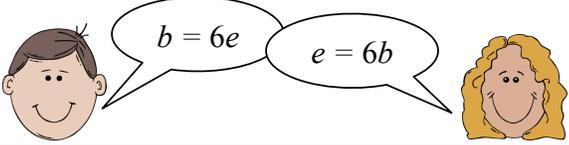
Slide P-4 shows two likely possible answers:

What is the equation? Why?

Let e be the **number** of eggs.
Let b be the **number** of egg boxes.
There are 6 eggs in each box.
Find an equation linking e and b .



$b = 6e$ $e = 6b$



Do you agree with the boy or with the girl? Why?

Students may think that the boy is right if they read the equation as ‘a box equals six eggs’. They may be quite surprised and confused to realize that in fact the girl is correct.

Suggest that students think of substituting some numbers for e and b :

Tell me two numbers for e and b that would make sense [e.g. $e=18, b=3$.]

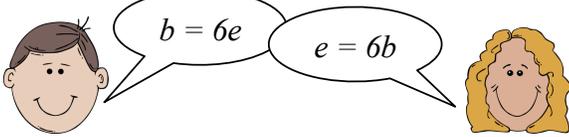
Which equation do these numbers fit?

Clearly here there will be $6 \times 3 = 18$ eggs, which matches the girl’s equation but not the boy’s.

Now show Slide P-5, which redefines the variables e and b :

What is the equation? Why?

Let e be the **cost** of an egg.
Let b be the **cost** of a box of eggs.
The price per egg is the same whether you buy them separately or in a box.
Find an equation linking e and b .



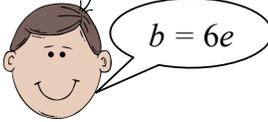
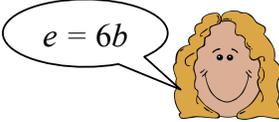
If they do not look too carefully, students may think that this is the same question as before.

What’s the same here and what’s different?

Students may assume that the boy’s equation must be correct this time, but encourage them to explain to each other why.

Slide P-6 may assist with comparing the two situations.

What's the same and what's different?

<p>Let e be the cost of an egg. Let b be the cost of a box of eggs.</p>  <p style="text-align: center;">$b = 6e$</p>	<p>Let e be the number of eggs. Let b be the number of egg boxes.</p>  <p style="text-align: center;">$e = 6b$</p>
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Can you explain why the equations are different in these two situations?

Why does each equation fit its situation?

Collaborative small-group work (25 minutes)

Ask students to work in groups of two or three.

Give each group some plain paper, a large sheet of paper, the cut-out definitions of the variables, and the cards from the *Equations* and *Statements* sheets.

I'd like you to match up the statements with the equations.

Sometimes more than one equation will go with a statement and sometimes more than one statement will go with an equation.

Write on the blank cards if there is a statement or equation that doesn't have a match.

Show the Slide *Working Together* (P-7), which explains how the students should work together:

Working Together

Take turns to:

1. Match an equation card with a sentence.
2. Explain your thinking to your partner.
3. Your partner must check and challenge your explanation if he or she disagrees.
4. Check if another card or another sentence matches as well.
5. Continue until you have matched all the cards.
6. Use the blank cards to write equations or sentences so that each card is grouped with at least one other card.

Students may use their blank paper for rough calculations and to explain their thinking to each other. While students are working you have two tasks: to note different student approaches to the task and to support student problem solving.

Note different student approaches

Listen to and watch students carefully. Notice how students make a start on the task, where they get stuck, and how they overcome any difficulties. Which expressions or equations do they find it easiest/hardest to make descriptions for? Where are their descriptions stronger/weaker? What misconceptions are manifested? What disagreements are common?

In particular, notice whether students are addressing the difficulties they experienced in the assessment task. Also note any common mistakes.

Support student problem solving

Help students to work together constructively. Remind them to look at the Slide for instructions on how to work. Check that students listen to each other and encourage them to explain their ideas using clear language. For example, students may be imprecise in describing the variables, saying ‘apples equals ...’ or ‘bananas are doubled’, etc.

*When you say ‘apples’, what **exactly** do you mean about apples? The weight of an apple?*

Say the equation in words. Start the sentence with ‘the cost of an apple ...’ Now make up another sentence, but this time start the sentence with ‘the cost of a banana ...’

Which has a bigger value x or y (a or b)?

Can the value of this variable be bigger than 5? Why/Why not?

Can the value of this variable be less than 1? Why/Why not?

Try not to solve students’ problems or do the reasoning for them. Instead, you might offer strategic advice or ask strategic questions to suggest ways of moving forward.

If you’re stuck with that card, you could put it to one side and place one of the others first.

Where students are interpreting the equations incorrectly, you could suggest that they try specific numbers.

Can you explain your thinking using some specific numbers for a and b ?

What values might a and b take?

What will happen if $a = 5$?

If a group of students finish placing all the cards and completing all the blank spaces, ask them to try to invent alternative descriptions for the same places in the grid.

You may like to ask students to visit other pairs and check their matches.

Once a group of students are satisfied with their matches, give them a glue stick so that they can stick the cards on to the large sheet of paper.

Whole-class discussion (15 minutes)

Conduct a whole-class discussion about what has been learned and explore the different descriptions that have been written. If you have noticed some interesting misconceptions as you circulated among the groups, you may want to focus the discussion on these.

Which equations did you find easiest/hardest? Why do you think that was?

Interpreting the equations, such as $ax = 5$, that combine two variables, are likely to be the most difficult.

What does the “ a ” represent here? What is x ? How do you know?

What words did you put for this one? Why?

Did anyone else do the same or something different? Which description do we prefer? Why?

Does anyone have a different way of explaining it?

Who agrees/disagrees? Why?

You may also want to ask them to use alternative ways to describe the relationship between the variables:

Describe in words the relationship between the two variables.

Now describe the same relationship in a different way.

Now describe this relationship using an equation that looks different from the ones on the cards.

You could also discuss equations such as $ay = 5$. Students may assume these types of equations are ‘impossible’.

Follow-up lesson: reviewing the assessment task (20 minutes)

Give each student a copy of the review task, *Real-life Equations (revisited)* and their original scripts from the assessment task, *Real-life Equations*. If you have not added questions to individual pieces of work then write your list of questions on the board. Students should select from this list only those questions they think are appropriate to their own work.

Look at your original responses and the questions [on the board/written on your script]. Answer these questions and revise your response.

*Now look at the new task sheet, *Real-life Equations (revisited)*. Can you use what you have learned to answer these questions?*

Some teachers give this as homework.

SOLUTIONS

Assessment task: Real-life Equations

- Equations (b) and (c) are correct.
These both express that the number of legs is four times the number of chairs.
- The first equation, $p = 2e$ means that he buys twice as many pens as erasers or half as many erasers as pens. The second equation, $x = 2y$, means that pens cost twice as much as erasers or that erasers are half the price of pens.

Collaborative task: Matching Equations and Statements

Suggested responses to be written on the blank cards are in **bold** type.

Apples cost half as much as bananas. Bananas cost twice as much as apples.	$y = 2x$ $x = \frac{1}{2}y$
I bought twice as many bananas as apples.	$b = 2a$ $a = \frac{1}{2}b$
Altogether I bought 5 apples and bananas.	$b+a = 5$
The apples I bought cost \$5.	$ax = 5$
I paid \$5 for all the apples and bananas I bought.	$ax+by = 5$
I bought 2 bananas	$b = 2$
Bananas cost half as much as apples.	$y = \frac{1}{2}x$ $x = 2y$
Bananas cost \$2	$y = 2$
Apples and bananas cost the same.	$x = y$

Assessment task: Real-life Equations (revisited)

- Equations (c) and (d) are correct, both expressing that the number of legs is eight times the number of spiders.
- The first equation, $c=2t$, means she sold twice as many cups of coffee as tea or half as many cups of tea as coffee.
The second equation, $tx + cy = 20$, means that Mary made \$20 from selling the tea and coffee.

Real-life Equations

1. Suppose that there are some chairs in a room and that each chair has 4 legs.

x = the **number** of chairs.

y = the **total number** of legs on all the chairs.

Put a check mark in the box next to every equation below that you think is correct.

(a) $x = 4y$

(b) $y = 4x$

(c) $x = \frac{y}{4}$

(d) $y = \frac{x}{4}$

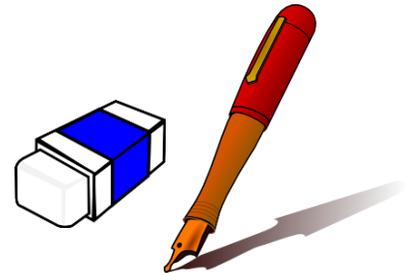
Explain your answer(s).

2. Suppose that Max buys some pens and erasers.

The cost of a pen in dollars is x and the cost of an eraser in dollars is y .

p = the **number** of pens he buys.

e = the **number** of erasers he buys.



The following two equations are true.

Write down the meaning of each equation in words.

(Refer to pens and erasers in your answers, don't just use letters.)

$p = 2e$	$x = 2y$
In everyday words this means:	In everyday words this means:
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Equations

a = the **number** of apples I bought

b = the **number** of bananas I bought

x = the **cost** of an apple in dollars

y = the **cost** of a banana in dollars

$y = 2x$	$b = 2a$
$a = \frac{b}{2}$	$x = 2y$
$ax = 5$	$ax + by = 5$
$y = \frac{x}{2}$	$b + a = 5$
$b = 2$	$x = \frac{y}{2}$

Statements

I bought twice as many bananas as apples.	The apples I bought cost \$5.
Altogether I bought 5 apples and bananas.	I paid \$5 for all the apples and bananas I bought.
Bananas cost twice as much as apples.	Bananas cost \$2.
Apples and bananas cost the same.	Apples cost half as much as bananas.

Real-life Equations (revisited)

1. Suppose that there are some spiders in a tank and each spider has 8 legs.

x = the **number** of spiders.

y = the **total number** of legs on all the spiders.

Put a check mark in the box next to every equation below that you think is correct.

(a) $x = 8y$

(b) $y = \frac{x}{8}$

(c) $x = \frac{y}{8}$

(d) $y = 8x$

Explain your answer(s).

.....

.....

.....

2. Suppose that Mary sells some cups of coffee and some cups of tea.
She sells a cup of tea for x dollars and a cup of coffee for y dollars.

t = the **number** of cups of tea she sold.

c = the **number** of cups of coffee she sold.



The following two equations are true.

Write down the meaning of each equation in everyday words.
(Refer to coffee and tea in your answers, don't just use letters.)

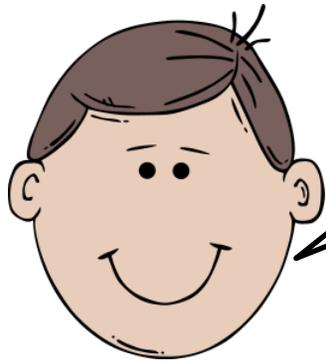
$c = 2t$	$tx + cy = 20$
In everyday words this means:	In everyday words this means:
.....
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Which is greater, x or y ? Why?

$$y = 4x$$

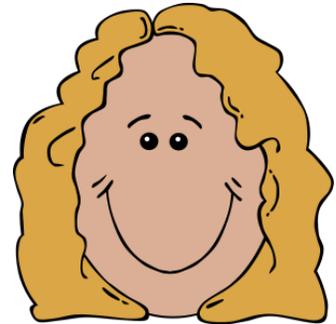
Which is greater, x or y ? Why?

$$y = 4x$$



x is greater because
it's multiplied by 4.

It depends what x
and y are.



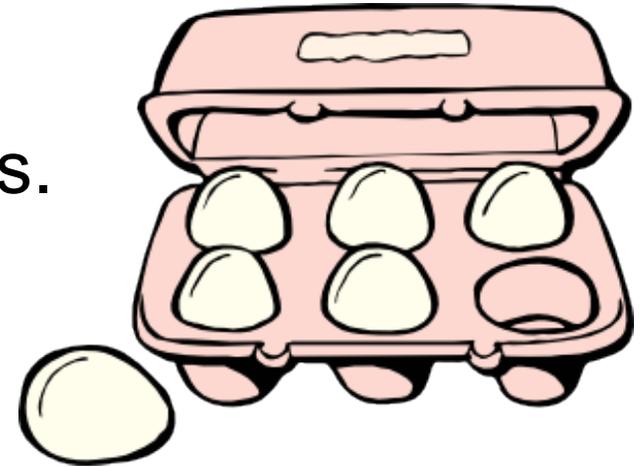
What is the equation? Why?

Let e be the **number** of eggs.

Let b be the **number** of egg boxes.

There are 6 eggs in each box.

Find an equation linking e and b .



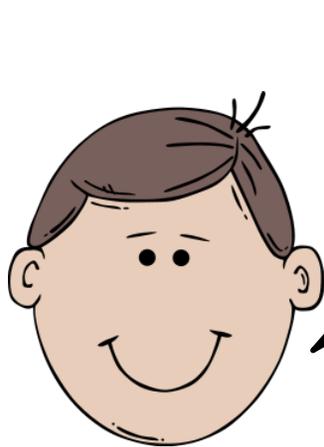
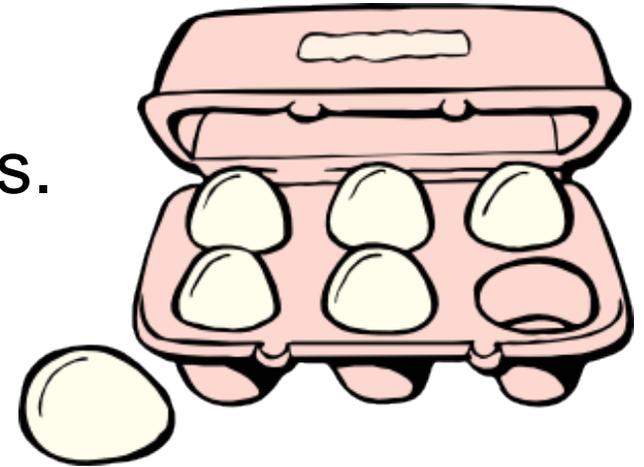
What is the equation? Why?

Let e be the **number** of eggs.

Let b be the **number** of egg boxes.

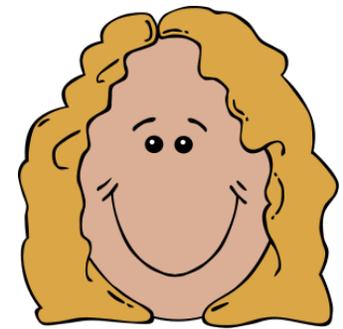
There are 6 eggs in each box.

Find an equation linking e and b .



$$b = 6e$$

$$e = 6b$$



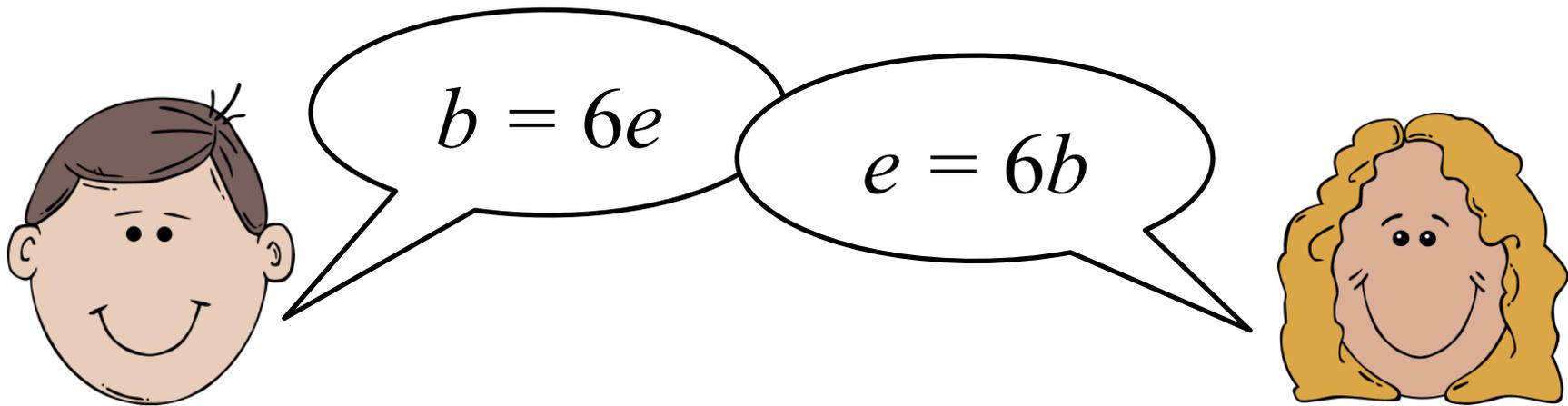
What is the equation? Why?

Let e be the **cost** of an egg.

Let b be the **cost** of a box of eggs.

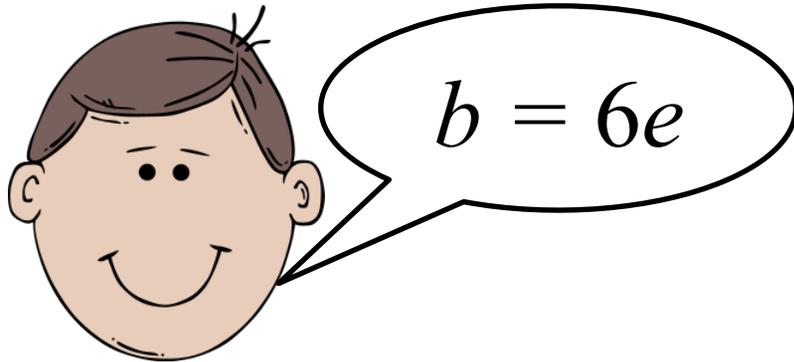
The price per egg is the same whether you buy them separately or in a box.

Find an equation linking e and b .

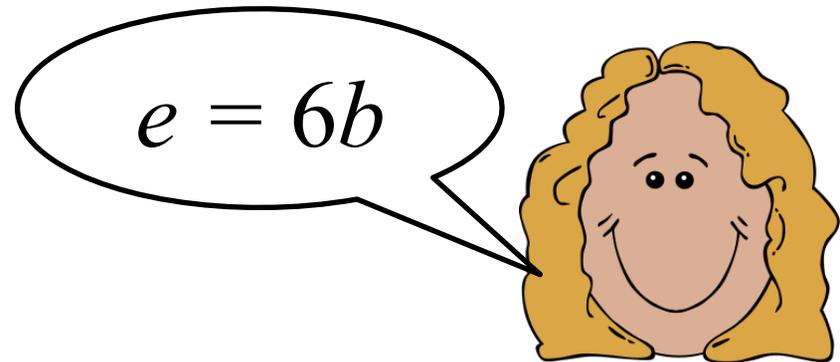


What's the same and what's different?

Let e be the **cost** of an egg.
Let b be the **cost** of a box of eggs.



Let e be the **number** of eggs.
Let b be the **number** of egg boxes.



Working Together

Take turns to:

1. Match an equation card with a sentence.
2. Explain your thinking to your partner.
3. Your partner must check and challenge your explanation if he or she disagrees.
4. Check if another card or another sentence matches as well.
5. Continue until you have matched all the cards.
6. Use the blank cards to write equations or sentences, so that each card is grouped with at least one other card.

Mathematics Assessment Project

Classroom Challenges

These materials were designed and developed by the
Shell Center Team at the Center for Research in Mathematical Education
University of Nottingham, England:

Malcolm Swan,
Nichola Clarke, Clare Dawson, Sheila Evans, Colin Foster, and Marie Joubert
with
Hugh Burkhardt, Rita Crust, Andy Noyes, and Daniel Pead

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Hugh Burkhardt, Daniel Pead, and Malcolm Swan at the University of Nottingham

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The full collection of Mathematics Assessment Project materials is available from

<http://map.mathshell.org>